

Dispersive properties of quasi-phase-matched optical parametric amplifiers

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Abstract

The dispersive properties of non-degenerate optical parametric amplification in quasi-phase-matched (QPM) nonlinear quadratic crystals with an arbitrary grating profile are theoretically investigated in the no-pump-depletion limit. The spectral group delay curve of the amplifier is shown to be *univocally* determined by its spectral power gain curve through a Hilbert transform. Such a constraint has important implications on the propagation of spectrally-narrow optical pulses through the amplifier. In particular, it is shown that anomalous transit times, corresponding to superluminal or even negative group velocities, are possible near local minima of the spectral gain curve. A possible experimental observation of such effects using a QPM Lithium-Niobate crystal is suggested.

PACS numbers: 42.65.Yj, 42.65.ky, 42.25.Bs

I. INTRODUCTION

The theoretical and experimental study of second-order nonlinear processes has received in the past years a renewed interest after the introduction of the quasi-phase-matching (QPM) technique [1, 2], which has lead to a major advance in applications such as frequency conversion, parametric oscillation and amplification, nonlinear optical frequency mixing and pulse shaping (see, for instance, [3, 4, 5] and references therein). The QPM technique uses a periodic modulation of the nonlinear $\chi^{(2)}$ coefficient (QPM grating) to compensate for refractive-index dispersion, permitting phase matching operation at any wavelength at room temperature, which would not be possible with birefringent phase matching. In addition, QPM enables the use of materials with strong nonlinearities which are not phase matchable by angle or temperature tuning, with a polarization state of interacting fields corresponding to the largest diagonal element of the $\chi^{(2)}$ tensor. Though the physics of the QPM technique is known since a long time [1], only with the recent technological advances in the electric-field poling of ferroelectric materials, such as LiNbO₃, LiTaO₃ and KTP, the experimental potentialities of QPM-based devices have become practicable and have reached nowadays a satisfactory degree of maturity [5]. The possibility of engineering a QPM grating by breaking its periodicity introduces additional degrees of freedom for light control which open the way for a novel class of devices [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. In particular, pulse compression and shaping by second harmonic generation (SHG) or by difference frequency generation (DFG) have been proposed and demonstrated using nonuniform QPM gratings [6, 7, 9, 12, 13]. In such cases, it was shown that in the low-conversion regime the spectral transfer function governing the frequency conversion process is related to the longitudinal QPM grating profile by a simple Fourier transform (for a detailed analysis and for a discussion of the distinctive aspects of SHG and DFG cases, see [12, 13]). The use of a QPM nonlinear crystal for parametric amplification of a signal field at frequency ω_1 by a strong pump field at frequency ω_3 has been demonstrated as well, both in waveguide and bulk geometries [16, 17], with the achievements of gains as large as 40 dB using nanosecond pump pulses [17]. As compared to most common parametric amplifiers based on birefringent phase matching, the use of a QPM-based parametric amplifier may offer the possibility of molding the spectral gain response of the amplifier by a suitable design of the QPM grating (see, e.g., [15]). Though the process of parametric amplification is accompanied by the generation of the

idler field at frequency $\omega_2 = \omega_3 - \omega_1$ and is thus equivalent to the DFG scheme recently studied in literature [13, 14], the simple Fourier analysis developed for DFG [13], which neglects amplification of the signal field, is inadequate to study the parametric amplification process in QPM gratings, and a more complex analysis is needed, based e.g. on matrix transfer techniques [15]. A general theoretical analysis of the properties of a QPM parametric amplifier, and especially of the interplay between its gain and dispersive properties, has not been pursued yet. Such a problem is closely related to the "inverse problem" of QPM grating synthesis, i.e. of the determination of a QPM grating profile that realizes a desired spectral gain response of the amplifier.

In this paper we provide a general analysis of the process of parametric amplification in an aperiodic QPM nonlinear crystal. We will show in a very general way that the spectral power gain curve of the amplifier univocally determines its dispersive properties, whereas the QPM grating profile that realizes such a spectral gain curve is not uniquely determined. The interplay between the spectral power gain and dispersive curves of the amplifier may lead to abnormal dispersive properties of the amplifier, such as the occurrence of superluminal or even negative group velocities similar to those observed in atomic amplifiers with a gain doublet [18, 19, 20]. The paper is organized as follows. In Sec.II the basic equations describing parametric pulse amplification in a QPM nonlinear crystal are reviewed. In the undepleted pump approximation, general properties of spectral gain and dispersion curves are derived in Sec.III, and the inverse problem of QPM synthesis is addressed. Section IV deals with the propagation of spectrally narrow pulses and analyzes the occurrence of anomalous regions corresponding to superluminal or even negative group velocities. Finally, in Sec.V the main conclusions are outlined.

II. PARAMETRIC AMPLIFICATION IN QPM QUADRATIC MEDIA: BASIC EQUATIONS

The starting point of the analysis is provided by the propagation scalar wave equation for a linearly-polarized electric field $\mathcal{E}(z, t)$ in a nonlinear $\chi^{(2)}$ medium with a quasiperiodic QPM grating profile. Under the usual plane-wave approximation and taking into account

material dispersion, from Maxwell's equations one can write (see, for instance, [21])

$$\frac{\partial^2 \mathcal{E}}{\partial z^2} + \int_{-\infty}^{\infty} d\omega k^2(\omega) \tilde{\mathcal{E}}(z, \omega) \exp(-i\omega t) = \mu_0 \frac{\partial^2 \mathcal{P}^{NL}}{\partial t^2}, \quad (1)$$

where $\tilde{\mathcal{E}}(z, \omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\omega \mathcal{E}(z, t) \exp(i\omega t)$ is the Fourier transform of $\mathcal{E}(z, t)$, $k(\omega) = (\omega/c_0)\sqrt{1 + \tilde{\chi}(\omega)} = (\omega/c_0)n(\omega)$ is the dispersion relation defined by the complex linear susceptibility $\tilde{\chi}(\omega)$ [or by the complex refractive index $n(\omega) = \sqrt{1 + \tilde{\chi}(\omega)}$], c_0 is the speed of light in vacuum, μ_0 is the vacuum magnetic permeability, and \mathcal{P}^{NL} is the nonlinear driving polarization term. For a quadratic medium and neglecting dispersion effects of second-order polarization, one can take $\mathcal{P}^{NL}(z, t) = \epsilon_0 \chi^{(2)}(z) \mathcal{E}^2(z, t)$, where $\chi^{(2)}$ is the spatially-modulated nonlinear susceptibility that accounts for the QPM grating. To study parametric amplification or DFG, we assume that $\mathcal{E}(z, t)$ is described by the superposition of three wavetrains with carrier frequencies ω_1 , ω_2 and $\omega_3 = \omega_1 + \omega_2$, corresponding to the signal, idler and pump waves, respectively. We thus set

$$\mathcal{E}(z, t) = \frac{1}{2} [\mathcal{E}_1(z, t) \exp(-i\omega_1 t) + \mathcal{E}_2(z, t) \exp(-i\omega_2 t) + \mathcal{E}_3(z, t) \exp(-i\omega_3 t) + c.c.] , \quad (2)$$

where the envelopes $\mathcal{E}_{1,2,3}(z, t)$ of wavetrains are assumed to vary slowly with respect to time t as compared to the exponential terms. In case of parametric amplification (or DFG), a strong pump wave and a weak signal wave are incident upon the nonlinear medium at the entrance plane $z = 0$, whereas the idler wave is generated by parametric interaction in the nonlinear medium and appears at the output plane $z = L$ (see Fig.1). Substitution of Eq.(2) into Eq.(1) and setting equal the terms oscillating at the same frequency, yields the following set of coupled-wave equations:

$$\frac{\partial^2 \mathcal{E}_1}{\partial z^2} + k^2(\omega_1 + i\partial_t) \mathcal{E}_1 = -\chi^{(2)} \left(\frac{\omega_1}{c_0} \right)^2 \mathcal{E}_3 \mathcal{E}_2^* \quad (3a)$$

$$\frac{\partial^2 \mathcal{E}_2}{\partial z^2} + k^2(\omega_2 + i\partial_t) \mathcal{E}_2 = -\chi^{(2)} \left(\frac{\omega_2}{c_0} \right)^2 \mathcal{E}_3 \mathcal{E}_1^* \quad (3b)$$

$$\frac{\partial^2 \mathcal{E}_3}{\partial z^2} + k^2(\omega_3 + i\partial_t) \mathcal{E}_3 = -\chi^{(2)} \left(\frac{\omega_3}{c_0} \right)^2 \mathcal{E}_1 \mathcal{E}_2 \quad (3c)$$

In deriving Eqs.(3), we have neglected the nonresonant terms in the nonlinear polarization driving term \mathcal{P}^{NL} , and we used the following property (see, for instance, [21]):

$$\int_{-\infty}^{\infty} d\omega k^2(\omega) \tilde{\mathcal{E}}(\omega) \exp(-i\omega t) = [k^2(\omega_0 + i\partial_t) \mathcal{A}(t)] \exp(-i\omega_0 t) \quad (4)$$

which is valid for any signal of the form $\mathcal{E}(t) = \mathcal{A}(t) \exp(-i\omega_0 t)$, where the operator $k^2(\omega_0 + i\partial_t)$ is defined through its power series expansion:

$$k^2(\omega_0 + i\partial_t) \equiv k^2(\omega_0) + 2ik(\omega_0) \left(\frac{\partial k}{\partial \omega} \right)_{\omega_0} \frac{\partial}{\partial t} - \left[\left(\frac{\partial k}{\partial \omega} \right)_{\omega_0}^2 + k(\omega_0) \left(\frac{\partial^2 k}{\partial \omega^2} \right)_{\omega_0} \right] \frac{\partial^2}{\partial t^2} + \dots \quad (5)$$

Notice that, in case of a wavetrain at carrier frequency ω_0 and slowly-varying envelope $\mathcal{A}(t)$, the power series on the right hand side in Eq.(5) may be truncated at some order. At leading order in the expansion one retrieves the wavenumber at the carrier frequency ω_0 , the second one determines the group-velocity $v_g = 1/(\partial k/\partial \omega)$ of the wavetrain, whereas higher order terms account for group velocity dispersion. If in Eqs.(3) the nonlinear terms are assumed to be weak and we truncate expansion (5) keeping the first term only, one sees that at leading order the amplitudes \mathcal{E}_l are oscillating in space like $\exp(\pm ik_l z)$, where $k_l \equiv k(\omega_l)$ ($l = 1, 2, 3$). If we consider only forward propagating waves, it is convenient to remove the fast oscillation over the wavelength spatial scale by setting $\mathcal{E}_l(z, t) = A_l(z, t) \exp(ik_l z)$. For the QPM grating, we assume that $\chi^{(2)}$ is a quasi-periodic function of z with period Λ , i.e. :

$$\chi^{(2)}(z) = \sum_{n=-\infty}^{\infty} \chi_n^{(2)}(z) \exp(-2in\pi z/\Lambda), \quad (6)$$

where the Fourier coefficients $\chi_n^{(2)}(z)$ are slowly varying functions of z over one period Λ . In practice, the slow dependence of coefficients on z can be achieved by a +/- reversal of domains in the ferroelectric crystal with a local period and local duty cycle that are slowly varying along the z axis (see, for instance, [12]). We assume that the nominal QPM period satisfies the phase matching condition:

$$\Lambda = M \frac{2\pi}{\Delta k}, \quad (7)$$

where $\Delta k \equiv k_3 - k_2 - k_1$ is the wavevector mismatch of interacting waves and M is a positive integer (QPM of order M). The evolution equations for the envelopes $A_l(z, t)$ ($l = 1, 2, 3$) can be derived, in the limit of weak nonlinearity and quasi-monochromatic wavetrains, by a multiple-scales asymptotic expansion (see, for instance, [22]); for the sake of completeness, a brief account of the derivation of the envelope equations is given in Appendix A. These

reads:

$$2ik_1 \frac{\partial A_1}{\partial z} = [k_1^2 - k^2(\omega_1 + i\partial_t)] A_1 - \frac{2k_1^2}{n_1^2} d_{eff} A_2^* A_3 \quad (8a)$$

$$2ik_2 \frac{\partial A_2}{\partial z} = [k_2^2 - k^2(\omega_2 + i\partial_t)] A_2 - \frac{2k_2^2}{n_2^2} d_{eff} A_1^* A_3 \quad (8b)$$

$$2ik_3 \frac{\partial A_3}{\partial z} = [k_3^2 - k^2(\omega_3 + i\partial_t)] A_3 - \frac{2k_3^2}{n_3^2} d_{eff}^* A_1 A_2 \quad (8c)$$

where:

$$d_{eff}(z) \equiv \frac{1}{2} \overline{\chi^{(2)}(z) \exp(i\Delta k z)} = \frac{1}{2} \chi_M^{(2)} \quad (9)$$

and the overline denotes a spatial average over a few modulation periods of the QPM grating. Equations (8) are the basic equations describing QPM parametric processes in the time domain. The linear operators on the right hand side of Eqs.(8), describing the linear dispersive and absorptive properties of the medium, may be expanded in power series [see Eq.(5)], and the number of terms that need to be kept depends on the the spectral extension of the wavetrains. In the following, we will consider spectral regions of transparency for the medium, so that we will neglect the imaginary part of the dispersion relation $k(\omega)$. In addition, for spectrally-narrow envelopes, only the leading-order term in the expansion may be kept, and Eqs.(8) reduce to the following ones:

$$\frac{\partial A_1}{\partial z} + \frac{1}{v_{g1}} \frac{\partial A_1}{\partial t} = i \frac{k_1}{n_1^2} d_{eff} A_2^* A_3 \quad (10a)$$

$$\frac{\partial A_2}{\partial z} + \frac{1}{v_{g2}} \frac{\partial A_2}{\partial t} = i \frac{k_2}{n_2^2} d_{eff} A_1^* A_3 \quad (10b)$$

$$\frac{\partial A_3}{\partial z} + \frac{1}{v_{g3}} \frac{\partial A_3}{\partial t} = i \frac{k_3}{n_3^2} d_{eff}^* A_1 A_2 \quad (10c)$$

where $v_{gl} = 1/(\partial k/\partial \omega)_{\omega_l}$ ($l = 1, 2, 3$) are the group velocities of the three wavetrains. For spectrally broad wavetrains, higher-order terms in the expansion (5) should be considered, which are responsible for group velocity dispersion and higher-order dispersion effects.

III. SPECTRAL GAIN CURVE AND GROUP DELAY ANALYSIS

In this section we provide a general analytical framework to determine the complex spectral gain curve of a QPM parametric amplifier in the limit of undepleted pump and derive a general relation among dispersive and gain properties of the amplifier. We assume that

the amplifier is pumped by a continuous-wave (CW) or quasi CW strong pump wave at frequency ω_3 , and a weak signal field at carrier frequency ω_1 , described by the envelope $A_1(0, t) = \int_{-\infty}^{\infty} \tilde{f}(\Omega) \exp(-i\Omega t)$, is incident upon the nonlinear crystal at the plane $z = 0$. In the limit of no pump depletion, which is valid for a low or moderate pump conversion, we may disregard Eq.(8c) and assume in Eqs.(8a) and (8b) a constant value of A_3 , given by $A_3 = [2I_3/(\epsilon_0 c_0 n_3)]^{1/2}$, where I_3 is the intensity of the incident CW pump and $n_3 = n(\omega_3)$. In this case, Eqs.(8a) and (8b) become linear equations, and the signal field envelope $A_1(L, t)$ at the exit of the medium can be written using standard Fourier analysis in the form:

$$A_1(L, t) = \int_{-\infty}^{\infty} d\Omega g(\Omega) \tilde{f}(\Omega) \exp(-i\Omega t) \quad (11)$$

where $\tilde{f}(\Omega)$ is the Fourier transform of the incident signal wavetrain and $g(\Omega)$ is the spectral gain curve of the amplifier. In order to determine the spectral gain curve $g(\Omega)$, let us search for a solution of Eqs.(8a) and (8b) in the form:

$$A_1(z, t) = u(z) \exp[-i\Omega t + i\beta(\Omega)z] \quad (12a)$$

$$A_2(z, t) = v^*(z) \frac{n_1}{n_2} \sqrt{\frac{k_2}{k_1}} \exp[i\Omega t - i\beta(\Omega)z], \quad (12b)$$

where Ω is the frequency offset from the reference carrier frequency ω_1 of the signal wave, $n_1 = n(\omega_1)$, $n_2 = n(\omega_2)$, and $\beta = \beta(\Omega)$ is defined by the relation:

$$\beta(\Omega) \equiv \frac{1}{4k_1} [k^2(\omega_1 + \Omega) - k_1^2] - \frac{1}{4k_2} [k^2(\omega_2 - \Omega) - k_2^2]. \quad (13)$$

The evolution equations of the complex envelopes $u(z)$ and $v(z)$ are obtained after inserting the Ansatz (12) into Eqs.(8). One then obtains for $u(z)$ and $v(z)$ the following coupled-mode equations:

$$\frac{du}{dz} = i\delta u + iq(z)v \quad (14a)$$

$$\frac{dv}{dz} = -i\delta v - iq^*(z)u \quad (14b)$$

where $\delta = \delta(\Omega)$ and $q(z)$ are give by:

$$\delta(\Omega) = \frac{1}{4k_1} [k^2(\omega_1 + \Omega) - k_1^2] + \frac{1}{4k_2} [k^2(\omega_2 - \Omega) - k_2^2], \quad (15)$$

$$q(z) = 2\pi d_{eff}(z) \sqrt{\frac{2I_3}{\epsilon_0 c_0 n_1 n_2 n_3 \lambda_1 \lambda_2}}. \quad (16)$$

In Eq.(16), λ_1 and λ_2 are the wavelengths in vacuum of signal and idler fields. Equations (14) have the form of the Zakharov-Shabat system encountered in problems of inverse scattering [23] and grating analysis [24, 25, 26, 27], the QPM grating profile $d_{eff}(z)$ playing the role of the scattering potential [see Eq.(16)]. The general solution to Eqs.(14) can be written as $(u(L, \delta), v(L, \delta))^T = \mathcal{M} (u(0, \delta), v(0, \delta))^T$, where the elements of the 2×2 transfer matrix $\mathcal{M} = \mathcal{M}(\delta)$ satisfy the conditions $\mathcal{M}_{22} = \mathcal{M}_{11}^*$, $\mathcal{M}_{21} = \mathcal{M}_{12}^*$, and $\det \mathcal{M} = \mathcal{M}_{11}\mathcal{M}_{22} - \mathcal{M}_{12}\mathcal{M}_{21} = 1$. From Eq.(12a), one realizes that the spectral gain curve $g(\Omega)$ of the amplifier is given by [28]:

$$g(\Omega) = \mathcal{M}_{11}(\Omega) \exp[i\beta(\Omega)L] \quad (17)$$

where the dependence of \mathcal{M}_{11} on Ω is through the function $\delta = \delta(\Omega)$ [see Eq.(15)]. The explicit calculation of $\mathcal{M}(\delta)$ for a generic scattering potential $q(z)$, i.e. QPM grating profile $d_{eff}(z)$, can be done numerically by a standard cascading technique in which the grating is decomposed as the cascade of successive uniform sections and \mathcal{M} is derived as the product of the elementary matrices of each uniform grating section (see, for instance, [26]). The application of the cascading technique for the study of DFG and parametric amplification in certain nonuniform QPM gratings has been recently addressed in [14, 15]. The determination of the spectral gain curve of a QPM parametric amplifier for a given QPM profile is thus easy: first, one has to compute the matrix element \mathcal{M}_{11} associated with the scattering problem [Eqs.(14)] by, e.g., the transfer matrix technique, for different values of frequency Ω ; the complex spectral gain curve is then retrieved by Eq.(17) with the use of Eq.(13). Here we are mainly concerned on investigating the interplay between amplitude and phase relationship of the complex spectral gain curve, which can be derived in a very general way using the results of the inverse scattering theory. To this aim, it is worth observing that, from inverse scattering theory, the scattering coefficient $a(\delta) \equiv \mathcal{M}_{11}(\delta) \exp(-i\delta L)$ is an analytic function in the lower part of the complex δ plane, i.e. for $\text{Im}(\delta) \leq 0$, $|a(\delta)| \geq 1$ on the real axis, and $a(\delta) \rightarrow 1$ as $\delta \rightarrow \infty$ [23]. In addition $a(\delta)$ has no zeros for $\text{Im}(\delta) \leq 0$, so that $1/a(\delta)$ belongs to the class of causal and minimal phase shift functions (see, for instance, [29]). This means that the knowledge of the modulus of $a(\delta)$ univocally determines its phase. In fact, since the function $F(\delta) = (\partial a / \partial \delta) / a = \partial(\ln a) / \partial \delta$ is analytic in the lower part of the complex δ plane and goes to zero for $\delta \rightarrow \infty$, the real and imaginary parts of $F(\delta)$ can

be related by a Hilbert transform, that is:

$$F(\delta) = \frac{1}{\pi i} \int_{-\infty}^{\infty} d\delta' \frac{F(\delta')}{\delta - \delta'}. \quad (18)$$

After setting $a(\delta) = |a(\delta)| \exp[i\phi_a(\delta)]$ and making equal the imaginary parts of both sides in Eq.(18), one obtains:

$$\frac{\partial \phi_a}{\partial \delta} = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\delta' \frac{\partial \ln |a(\delta')|}{\partial \delta'} \frac{1}{\delta - \delta'} \quad (19)$$

In order to further proceed in the analysis, we must specify the dependence of β and δ on Ω . From Eqs.(13) and (15), the expansion of $k = k(\omega)$ in power series of Ω yields:

$$\beta(\Omega) = \frac{1}{2}\Omega \left(\frac{1}{v_{g1}} + \frac{1}{v_{g2}} \right) + \frac{1}{4}\Omega^2 \left(k_1'' - k_2'' + \frac{k_1'^2}{k_1} - \frac{k_2'^2}{k_2} \right) + \dots \quad (20)$$

$$\delta(\Omega) = \frac{1}{2}\Omega \left(\frac{1}{v_{g1}} - \frac{1}{v_{g2}} \right) + \frac{1}{4}\Omega^2 \left(k_1'' + k_2'' + \frac{k_1'^2}{k_1} + \frac{k_2'^2}{k_2} \right) + \dots \quad (21)$$

where $k_{1,2}'$ and $k_{1,2}''$ are the first and second derivatives of $k(\omega)$ evaluated at the frequencies $\omega_{1,2}$. For the sake of simplicity, let us assume that $v_{g1} \neq v_{g2}$, which is usually the case for a non-degenerate interaction, and that the bandwidth of the amplifier is narrow enough to neglect group-velocity dispersion and higher-order dispersive effects in Eqs.(20) and (21). At leading order we may hence assume:

$$\delta(\Omega) = \frac{1}{2}\Omega \left(\frac{1}{v_{g1}} - \frac{1}{v_{g2}} \right) \quad , \quad \beta(\Omega) = \frac{1}{2}\Omega \left(\frac{1}{v_{g1}} + \frac{1}{v_{g2}} \right) \quad (22)$$

If we introduce the spectral power gain curve $G(\Omega)$ and group delay $\tau_g(\Omega)$ of the amplifier, defined as $G(\Omega) \equiv |g(\Omega)|^2$ and $\tau_g(\Omega) \equiv \partial \phi_g / \partial \Omega$, where $\phi_g(\Omega)$ is the phase of $g(\Omega)$, from Eqs.(17), (19) and (22), and recalling that $a(\delta) = \mathcal{M}_{11}(\delta) \exp(-i\delta L)$, one finally obtains the following relationship between the power spectral gain G and group delay τ_g of the amplifier:

$$\tau_g = \frac{L}{v_{g1}} + \frac{1}{2} \left(\frac{1}{v_{g1}} - \frac{1}{v_{g2}} \right) \frac{1}{\pi} \int_{-\infty}^{\infty} d\delta' \frac{\partial \ln \sqrt{G(\delta')}}{\partial \delta'} \frac{1}{\delta' - \delta} \quad (23)$$

that is:

$$\tau_g = \frac{L}{v_{g1}} \pm \frac{1}{\pi} \int_{-\infty}^{\infty} d\Omega' \frac{\partial \ln \sqrt{G(\Omega')}}{\partial \Omega'} \frac{1}{\Omega' - \Omega} \quad (24)$$

where the upper [lower] sign occurs if $v_{g1} < v_{g2}$ [$v_{g1} > v_{g2}$]. Equation (24) shows that the power spectral gain curve $G(\Omega)$ of the amplifier uniquely determines its phase response,

i.e. the group delay τ_g . The constraint imposed by Eq.(24) has important impacts on the response of the amplifier, reducing the flexibility in designing a given spectral gain curve as compared to the related DFG problem [13]. For instance, the possibility of "squaring" the spectral power gain curve of the amplifier comes at the price of an increased phase distortion [30]. Such limitations come out just because the spectral gain curve satisfies the minimum-phase condition, which is not the case of the DFG transfer function [28]. Another interesting consequence of Eq.(24) is the existence of spectral regions where the group delay becomes superluminal or even negative, which will be studied in detail in the next section. Here we end our general analysis by briefly addressing the issue of the synthesis of the spectral gain curve $g(\Omega)$ of a QPM amplifier (the so-called "inverse problem"), that is the determination of the QPM profile $d_{eff}(z)$ that realizes a target spectral response $g(\Omega)$. Such a problem has been extensively studied on several occasions in the context of optical filters and gratings theory (see, for instance, [31] and references therein), and therefore it will be briefly quoted here. First of all, let us recall from inverse scattering theory that in Eq.(14) the scattering potential $q(z)$ is uniquely determined from the knowledge of $r(\delta) = \mathcal{M}_{12}/\mathcal{M}_{11}$, which must satisfy minimal requirements of causality, and that numerical techniques for determination of $q(z)$ from the knowledge of $r(\delta)$ are well developed [31, 32]. Let us now assign a target power spectral gain curve $G(\Omega)$, and ask if there is a QPM grating profile $d_{eff}(z)$ that realizes such a gain response. First of all, $G(\Omega)$ should satisfy minimal conditions: $G(\Omega) \geq 1$ for all real frequencies Ω , be analytic in the lower [upper] part of the complex Ω plane if $v_{g1} < v_{g2}$ [$v_{g1} > v_{g2}$], $G(\Omega) \rightarrow 1$ for $\Omega \rightarrow \infty$, and the integral $\int_0^\infty d\Omega \ln G(\Omega)/(1 + \Omega^2)$ should not diverge. In this case, the scattering coefficient $a(\delta)$, and hence the matrix coefficient \mathcal{M}_{11} , can be univocally determined owing to the minimal phase shift condition through Eq.(19). Once $\mathcal{M}_{11}(\delta)$ has been calculated from $G(\Omega)$, the modulus of $r(\delta)$ is determined by $|r(\delta)|^2 = (1 - 1/|\mathcal{M}_{11}|^2)$, however some degrees of freedom are left in the phase of $r(\delta)$, yet ensuring the requirements of causality (see Sec.4 of Ref.[29]). By changing the phase of $r(\delta)$, one basically changes the spectral *phase* response of the generated idler wave, which is related to the phase of \mathcal{M}_{12} (see [28]), without affecting the spectral gain amplitude and phase at the signal wavelength. For a physically realizable power spectral gain curve $G(\Omega)$, there is not, hence, a unique QPM profile that realizes such a gain response, the different QPM profiles corresponding to different spectral phase response of the generated idler wave. The non-uniqueness of the inverse problem is analogous to that encountered, e.g., in the

design of fiber Bragg gratings for use in transmission [33, 34].

IV. ANOMALOUS PROPAGATION OF SPECTRALLY NARROW OPTICAL PULSES: SUPERLUMINAL AND NEGATIVE TRANSIT TIMES

A physically important consequence of the dispersive properties of a QPM parametric amplifier, stated by Eq.(24), is the existence of abnormal group delays, that is the existence of spectral regions where a spectrally-narrow signal pulse can travel through the amplifier with a superluminal ($\tau_g < L/c_0$) or even negative ($\tau_g < 0$) transit time [35]. Let us consider the propagation of a spectrally-narrow signal pulse through the QPM amplifier. We assume that the spectrum $\tilde{f}(\Omega)$ of the incident pulse is centered at a frequency Ω_0 , so that we can write $A_1(0, t) = \exp(-i\Omega_0 t)h_0(t)$ and $A_1(L, t) = \exp(-i\Omega_0 t)h_L(t)$ at the input and output planes of the amplifier. If we now assume that the spectral extent of the envelope $h_0(t)$ is narrow enough such that the gain curve $g(\Omega)$ of the amplifier varies slowly over the pulse bandwidth, an asymptotic expansion for $h_L(t)$ may be obtained, as detailed in Appendix B. At leading order one obtains:

$$h_L(t) \simeq g_0 h_0(t - \Delta\tau) \quad (25)$$

where $g_0 = g(\Omega_0)$ is the gain of the amplifier at the pulse central frequency Ω_0 and $\Delta\tau$ is the complex group delay time, which is given by (see Appendix B):

$$\Delta\tau \equiv -i \left(\frac{\partial \ln g}{\partial \Omega} \right)_{\Omega_0} = \tau_g(\Omega_0) - i \left(\frac{\partial \ln \sqrt{G}}{\partial \Omega} \right)_{\Omega_0}. \quad (26)$$

Notice that, since the group delay $\Delta\tau$ is in general complex-valued, even at leading order in the expansion [see Eq.(B7) in Appendix B] the amplifier produces a pulse distortion [36]. However, if Ω_0 is a stationary point of the spectral gain curve, then the complex group delay $\Delta\tau$ becomes real-valued and equal to the usual group delay $\tau_g = \partial \phi_g / \partial \Omega$. In this case, at leading order the amplifier is distortionless and the pulse transit time is related to the spectral gain curve $G(\Omega)$ by means of the Hilbert transform given in Eq.(24). Let us now suppose that $v_{g1} > v_{g2}$ and that Ω_0 is a maximum of the spectral power gain curve; then for $\Omega \simeq \Omega_0$, the contribution to the integral on the right hand side in Eq.(24) near the singularity $\Omega' = \Omega$ is negative, and the integral is thus expected to be negative. This implies that $\tau_g > L/v_{g1}$, i.e. the effective group velocity of the pulse is lower than v_{g1} . Conversely,

if Ω_0 is a minimum of the spectral power gain curve, the integral in Eq.(23) is expected to be positive, and the transit time τ_g lower than L/v_{g1} . This means that superluminal or even negative transit times may be expected whenever the pulse spectrum is centered around a minimum of the spectral gain curve [37]. The existence of anomalous transit times for a pulse tuned midway two amplification peaks is indeed a rather general result, and previous demonstrations of negative transit times have been reported for optical pulses as well as for electronic signals using a gain-doublet amplifier [18, 19, 20, 38].

To get a quantitative analysis of such effects, let us refer to a periodically-poled Lithium Niobate (PPLN) crystal pumped at the wavelength $\lambda_3 = 532$ nm with a signal field at $\lambda_1 = 1.55$ μm . The idler wave corresponds to the wavelength $\lambda_2 = 810$ nm; we further assume extraordinary wave propagation, so that $d = d_{33} \simeq 27$ pm/V. The temperature-dependent dispersion relation $k = k(\omega) = n(\omega)\omega/c_0$ for extraordinary waves in Lithium Niobate is determined using Sellmeier equations from Ref.[39]. At 25°C, one can estimate $v_{g1} \simeq 0.4815c_0$, $v_{g2} \simeq 0.44220c_0$, and a first-order QPM period $\Lambda = 7.39$ μm , which is accessible with current poling technology. The main task in the design of the QPM grating is to realize a gain curve $G(\Omega)$ with a local minimum, e.g. at $\Omega_0 = 0$. There are several possibilities to achieve a spectral dip in the gain curve of a QPM amplifier, such as the inclusion of a defect in an otherwise periodic QPM grating or the cascading of two periodic QPM gratings. We consider here the latter case, however a similar analysis could be done for the former configuration. The QPM grating, shown in Fig.2, consists of a sequence of two $+/-$ square-wave uniform gratings, each of length a and period $\Lambda = 2\pi/\Delta k$ (first-order QPM), separated by a distance l . For such a structure one can easily calculate d_{eff} , and hence $q(z)$, obtaining $q(z) = q_0$ for $0 < z < a$, $q(z) = 0$ for $a < z < a + l$, and $q(z) = q_0 \exp(i\Phi)$ for $a + l < z < 2a + l$, where $q_0 \equiv (2/\pi)d_{33}[8\pi^2 I_3/(\epsilon_0 c_0 n_1 n_2 n_3 \lambda_1 \lambda_2)]^{1/2}$ and Φ is a phase shift that depends on the relative phases of the two square waves in the two grating sections. For instance, the reversal of sign of d_{eff} , corresponding to $\Phi = \pi$, occurs if the two square waveforms in the two grating sections are shifted each other by half a period [see Fig.2(b)]. Notice that, in the limiting case $l \rightarrow 0$, the structure reduces to a periodic QPM grating of length $2a$ with a defect at the center of the structure. In order to calculate the spectral gain curve $g(\Omega)$ of the amplifier, we need to evaluate the matrix element \mathcal{M}_{11} according to Eq.(17). The transfer matrix of the structure shown in Fig.2 can be readily calculated as the product of the three transfer matrices corresponding to the propagation

in the first uniform grating section, the middle grating-free section, and the second uniform grating section in the reverse order, i.e.:

$$\mathcal{M} = \begin{pmatrix} \cosh(\theta a) + i\frac{\delta}{\theta} \sinh(\theta a) & i \exp(i\Phi) \frac{q_0}{\theta} \sinh(\theta a) \\ -i \exp(-i\Phi) \frac{q_0}{\theta} \sinh(\theta a) & \cosh(\theta a) - i\frac{\delta}{\theta} \sinh(\theta a) \end{pmatrix} \times \begin{pmatrix} \exp(i\delta l) & 0 \\ 0 & \exp(-i\delta l) \end{pmatrix} \times \begin{pmatrix} \cosh(\theta a) + i\frac{\delta}{\theta} \sinh(\theta a) & i\frac{q_0}{\theta} \sinh(\theta a) \\ -i\frac{q_0}{\theta} \sinh(\theta a) & \cosh(\theta a) - i\frac{\delta}{\theta} \sinh(\theta a) \end{pmatrix} \quad (27)$$

where $\theta \equiv \sqrt{|q_0|^2 - \delta^2}$. From Eqs.(17) and (27) one readily obtains:

$$g(\Omega) = \exp(i\delta l + i\beta L) \left[\cosh^2(\theta a) - \frac{\delta^2}{\theta^2} \sinh^2(\theta a) + i\frac{\delta}{\theta} \sinh(2\theta a) + \exp(i\Phi) \frac{|q_0|^2}{\theta^2} \sinh^2(\theta a) \exp(-2i\delta l) \right] \quad (28)$$

where $\beta = \beta(\Omega)$ and $\delta = \delta(\Omega)$ are given, in the general case, by Eqs.(13) and (15). Around the resonance $\Omega = 0$, the approximate equations (22) may be used, which require the knowledge of the group velocities for signal and idler fields solely. A typical behavior of power spectral gain $G(\Omega)$ and corresponding group delay $\tau_g(\Omega)$ for a few different values of Φ are shown in Figs.3,4 and 5. The parameter values of the QPM grating structure are $a = 3$ mm and $l = 2$ mm. Notice that, in correspondence of local minima in the power spectral gain curve, the group delay curve shows local minima, corresponding to superluminal and even negative group delays (see, for instance, Fig.3). An inspection of Eq.(28) reveals that, for $\Phi = \pi$, the curve $G(\Omega) = |g(\Omega)|^2$ shows a dip at $\Omega = 0$, with $G(0) = 1$ and with a corresponding minimum in the group delay curve which can be calculated analytically in a closed form and reads:

$$\tau_g(\Omega = 0) = \frac{L}{2} \left(\frac{1}{v_{g1}} - \frac{1}{v_{g2}} \right) - \frac{l}{2} \left(\frac{1}{v_{g2}} - \frac{1}{v_{g1}} \right) \times \left[\cosh^2(|q_0|a) + \sinh^2(|q_0|a) + \frac{\sinh(2|q_0|a)}{|q_0|l} \right] \quad (29)$$

For $\Phi \neq \pi$, the curves $G(\Omega)$ and $\tau_g(\Omega)$ may become strongly asymmetric (as in Fig.4), the minimum in the gain curve is shifted away from $\Omega = 0$ or multiple local minima may occurs; however the general rule of faster than v_{g1} group velocity near local minima of the power spectral gain curve holds. Figure 6 shows the behavior of the group delay at $\Omega = 0$, for the case of perfect phase reversal of the two grating sections (i.e., for $\Phi = \pi$), versus intensity of the pump wave. The figure clearly indicates the possibility of controlling the transit time of a spectrally-narrow signal pulse by the pump intensity. Notice that the group

delay becomes superluminal, i.e. smaller than L/c_0 , at $I_3 \simeq 105 \text{ MW/cm}^2$ and negative at $I_3 \simeq 135 \text{ MW/cm}^2$. To reach the latter regime, the pump intensity corresponds to an off-resonance power gain peak as large as 40 dB [see Fig.3(a)]. However, since the pulse spectrum is centered at $\Omega = 0$ where $G(\Omega) \simeq 1$, *the main effect of the amplifier on pulse propagation near the gain dip is to advance the pulse in time, not to amplify it*. A similar effect was predicted and observed for pulse propagation in an atomic gain-doublet amplifier [19, 20]. What physically happens is that in the first section of the QPM grating the signal pulse is amplified, with the generation of the idler wave ($\omega_3 \rightarrow \omega_1 + \omega_2$); however owing to the phase reversal in the second QPM grating, a back conversion process ($\omega_1 + \omega_2 \rightarrow \omega_3$) occurs in the second grating section. The result of such a cascading process is that the signal pulse is basically not amplified at the output of the crystal, however the phase delays suffered by its spectral components, as ruled by the causality condition [Eq.(24)], produce a temporal advancement with no appreciable pulse distortion. We have checked the prediction of pulse propagation based on the group delay analysis by direct numerical simulation of pulse propagation in a two-section QPM PPLN crystal with perfect phase reversal ($\Phi = \pi$) starting from Eqs.(8) in the limit of an undepleted pump. As a probing pulse, we assumed a transform-limited Gaussian pulse, tuned at $\Omega = 0$, with a pulse duration (FWHM) of 250 ps to ensure the spectrally-narrow pulse limit. Figure 7 shows the traces of the incident signal pulse at the entrance plane of the crystal $z = 0$ (dotted line) and of transmitted pulses at the exit plane $z = L$ (solid lines) for increasing values of the pump intensity I_3 ; the pulse intensity of the output waveforms are normalized to the peak intensity of incident pulse. It is remarkable that, at pump intensities corresponding to, e.g., curve 4, the transmitted pulse leaves the amplifier *before* the peak of Gaussian incident pulse has entered into the crystal. To make a quantitative estimate of pump power levels required to observe such effects, let us assume a Gaussian pump with a beam waist of $\simeq 200 \text{ }\mu\text{m}$; then curve 3 of Fig.7 corresponds to a pump peak power of $\simeq 170 \text{ kW}$. Using a pulsed pump of duration (FWHM) of $\sim 5 \text{ ns}$, i.e. about twenty times longer than the probing pulses, a pump pulse energy of $\simeq 0.90 \text{ mJ}$ is required, which can be obtained using a frequency-doubled Q-switched Nd-based laser system as a pump source.

V. CONCLUSIONS

In this paper we have presented a general analysis of the dispersive properties of QPM optical parametric amplifiers. In the limit of no pump depletion, the parametric interaction of idler and signal fields, that accounts for material dispersion at any order, may be described by coupled-mode equations which have a canonical form widely encountered in problems of inverse scattering and optical gratings and filters design. One of the main consequences of the analysis is that the spectral power gain curve of the amplifier defines univocally its dispersive curve through a Hilbert-like transform. This circumstance may be of major relevance in the design and synthesis of QPM amplifiers for applications in pulse shaping and control, and imposes unavoidable physical limits to the realization of dispersionless amplifiers with a flat spectral gain [30]. Our analysis also demonstrates that engineered QPM grating profiles can simulate dispersive properties of resonant pulse propagation in inverted atomic media. In particular, in this work we have proposed and studied the possibility of observing and controlling superluminal and negative group velocities of picosecond optical pulses in a PPLN amplifier, which shows features similar to those found in atomic amplifiers with a gain doublet [18, 19, 20].

APPENDIX A: DERIVATION OF ENVELOPE EQUATIONS

In this Appendix we derive the envelope equations [Eqs.(8a)-(8c)] given in the text by a multiple scale asymptotic analysis of the original equations [Eqs.(3a)-(3c)]. Such an analysis assumes that the nonlinearity of the medium is weak and that the spectral extent of the interacting fields is narrow enough such that in Eq.(5) the operator $k^2(\omega_0 + i\partial_t) - k^2(\omega_0)$ can be treated as a perturbation term. With these assumptions, it is worth rewriting Eqs.(3a)-(3c) in the form:

$$\frac{\partial^2 \mathcal{E}_1}{\partial z^2} + k_1^2 \mathcal{E}_1 = \epsilon \left\{ \left[k_1^2 - k^2(\omega_1 + i\partial_t) \right] \mathcal{E}_1 - \chi^{(2)} \left(\frac{\omega_1}{c_0} \right)^2 \mathcal{E}_3 \mathcal{E}_2^* \right\} \quad (\text{A1})$$

$$\frac{\partial^2 \mathcal{E}_2}{\partial z^2} + k_2^2 \mathcal{E}_2 = \epsilon \left\{ \left[k_2^2 - k^2(\omega_2 + i\partial_t) \right] \mathcal{E}_2 - \chi^{(2)} \left(\frac{\omega_2}{c_0} \right)^2 \mathcal{E}_3 \mathcal{E}_1^* \right\} \quad (\text{A2})$$

$$\frac{\partial^2 \mathcal{E}_3}{\partial z^2} + k_3^2 \mathcal{E}_3 = \epsilon \left\{ \left[k_3^2 - k^2(\omega_3 + i\partial_t) \right] \mathcal{E}_3 - \chi^{(2)} \left(\frac{\omega_3}{c_0} \right)^2 \mathcal{E}_1 \mathcal{E}_2 \right\} \quad (\text{A3})$$

where ϵ is a small parameter that defines the order of magnitude of the perturbation terms entering on the right hand side in Eqs.(A1)-(A3). The problem is to construct an asymptotic solution of the perturbed equations as $\epsilon \rightarrow 0$ which is valid uniformly with respect to the spatial variable z . Therefore we look for a solution to Eqs.(A1)-(A3) in the form:

$$\mathcal{E}_l = \mathcal{E}_l^{(0)} + \epsilon \mathcal{E}_l^{(1)} + \epsilon^2 \mathcal{E}_l^{(2)} + \dots \quad (\text{A4})$$

($l = 1, 2, 3$), and we require that the asymptotic expansion be uniformly valid. This condition can be satisfied by introducing multiple scales for z , i.e. by assuming that \mathcal{E}_l depends on Z_0, Z_1, Z_2, \dots , where $Z_0 = z, Z_1 = \epsilon z, Z_2 = \epsilon^2 z, \dots$. The introduction of multiple spatial scales is fundamental to remove secular growing terms that arise in the perturbation expansion. Introducing expansion (A4) into Eqs.(A1)-(A3), using the derivative rule $\partial_z^2 = \partial_{Z_0}^2 + \epsilon(\partial_{Z_0}\partial_{Z_1} + \partial_{Z_1}\partial_{Z_0}) + \dots$ and collecting the terms of the same order in the equations so obtained, a hierarchy of equations for successive corrections to \mathcal{E}_l is obtained. At leading order, $O(\epsilon^0)$, one obtains:

$$\frac{\partial^2 \mathcal{E}_l^{(0)}}{\partial Z_0^2} + k_l^2 \mathcal{E}_l^{(0)} = 0. \quad (\text{A5})$$

($l = 1, 2, 3$). If we consider forward propagating waves, i.e. a copropagating interaction, the solutions to Eq.(A5) are given by:

$$\mathcal{E}_l^{(0)}(Z_0, Z_1, \dots; t) = A_l(Z_1, \dots; t) \exp(ik_l Z_0), \quad (\text{A6})$$

where the amplitudes A_l depend on time and on slow spatial variables. At $O(\epsilon)$ one obtains:

$$\frac{\partial^2 \mathcal{E}_l^{(1)}}{\partial Z_0^2} + k_l^2 \mathcal{E}_l^{(1)} = G_l^{(1)} \quad (\text{A7})$$

where the driving terms $G_l^{(1)}$ in Eqs.(A7) are given by:

$$G_1^{(1)} = -2\partial_{Z_0}\partial_{Z_1}\mathcal{E}_1^{(0)} + [k_1^2 - k^2(\omega_1 + i\partial_t)]\mathcal{E}_1^{(0)} - \chi^{(2)}\left(\frac{\omega_1}{c_0}\right)^2 \mathcal{E}_3^{(0)}\mathcal{E}_2^{(0)*} \quad (\text{A8})$$

$$G_2^{(1)} = -2\partial_{Z_0}\partial_{Z_1}\mathcal{E}_2^{(0)} + [k_2^2 - k^2(\omega_2 + i\partial_t)]\mathcal{E}_2^{(0)} - \chi^{(2)}\left(\frac{\omega_2}{c_0}\right)^2 \mathcal{E}_3^{(0)}\mathcal{E}_1^{(0)*} \quad (\text{A9})$$

$$G_3^{(1)} = -2\partial_{Z_0}\partial_{Z_1}\mathcal{E}_3^{(0)} + [k_3^2 - k^2(\omega_3 + i\partial_t)]\mathcal{E}_3^{(0)} - \chi^{(2)}\left(\frac{\omega_3}{c_0}\right)^2 \mathcal{E}_1^{(0)}\mathcal{E}_2^{(0)}. \quad (\text{A10})$$

To further proceed, let us assume that $\chi^{(2)}(z)$ is a quasi-periodic function of z with a period Λ satisfying the QPM condition given by Eq.(7) in the text, and that Λ is much smaller than the crystal length and of the same order of magnitude as the wavelengths of interacting fields. It is then worth separating the fast and slow dependence of $\chi^{(2)}$ on z by setting:

$$\chi^{(2)}(z) = \sum_{n=-\infty}^{\infty} \chi_n^{(2)}(Z_1) \exp(-2\pi i n Z_0 / \Lambda) \quad (\text{A11})$$

where the coefficients $\chi_n^{(2)}$ of the Fourier series are allowed to vary on the slow spatial scale Z_1 and satisfy the condition $\chi_{-n}^{(2)} = \chi_n^{(2)*}$. The solutions to Eqs. (A7) are bounded with respect to Z_0 provided that the driving term $G_l^{(1)}$ does not contain terms oscillating like $\exp(ik_l Z_0)$. The solvability conditions allow one to derive the evolution equations of the envelopes A_l on the slow spatial scale Z_1 . Substitution of Eqs.(A6) and (A11) into Eqs.(A8)-(A10), after collecting the the terms oscillating like $\exp(ik_l Z_0)$, one obtains the following solvability conditions:

$$-2ik_1 \frac{\partial A_1}{\partial Z_1} + [k_1^2 - k^2(\omega_1 + i\partial_t)] A_1 - \chi_M^{(2)} \left(\frac{\omega_1}{c_0} \right)^2 A_2^* A_3 = 0 \quad (\text{A12})$$

$$-2ik_2 \frac{\partial A_2}{\partial Z_1} + [k_2^2 - k^2(\omega_2 + i\partial_t)] A_2 - \chi_M^{(2)} \left(\frac{\omega_2}{c_0} \right)^2 A_1^* A_3 = 0 \quad (\text{A13})$$

$$-2ik_3 \frac{\partial A_3}{\partial Z_1} + [k_3^2 - k^2(\omega_3 + i\partial_t)] A_3 - \chi_M^{(2)*} \left(\frac{\omega_3}{c_0} \right)^2 A_1 A_2 = 0 \quad (\text{A14})$$

If we stop the asymptotic expansion at order $\sim \epsilon$, re-introducing the original spatial variable z instead of Z_1 in Eqs.(A12)-(A14) and setting $\epsilon = 1$, one finally obtains Eqs.(8) given in the text, where d_{eff} is defined through Eq.(9).

APPENDIX B: PROPAGATION OF SPECTRALLY-NARROW OPTICAL PULSES

Let $A_1(0, t) = \exp(-i\Omega_0 t) h_0(t)$ be the signal field envelope incident upon the amplifier with a frequency offset Ω_0 from the carrier ω_1 . From Eq.(11) the pulse waveform at the exit of the amplifier can be written as $A_1(L, t) = h_L(t) \exp(-i\Omega_0 t)$, where:

$$h_L(t) = g(\Omega_0 + i\partial_t) h_0(t) \quad (\text{B1})$$

and the operator on the right hand side in Eq.(B1) is defined, as usual, by its power series expansion [21]. We now assume that the spectral extent of the envelope $h_0(t)$ is narrow enough such that the gain curve $g(\Omega)$ of the amplifier varies slowly over the pulse bandwidth. One can then expand $g(\Omega)$ in power series around $\Omega = \Omega_0$ as follows:

$$g(\Omega) = \exp[\ln g(\Omega)] = g_0 \exp \left[\left(\frac{\partial \ln g}{\partial \Omega} \right)_{\Omega_0} (\Omega - \Omega_0) + \frac{1}{2} \left(\frac{\partial^2 \ln g}{\partial \Omega^2} \right)_{\Omega_0} (\Omega - \Omega_0)^2 + \dots \right] = g_0 \mathcal{B} \exp \left[\left(\frac{\partial \ln g}{\partial \Omega} \right)_{\Omega_0} (\Omega - \Omega_0) \right] \quad (\text{B2})$$

where we have set:

$$\mathcal{B}(\Omega - \Omega_0) = \exp \left[\frac{1}{2} \left(\frac{\partial^2 \ln g}{\partial \Omega^2} \right)_{\Omega_0} (\Omega - \Omega_0)^2 + \dots \right]. \quad (\text{B3})$$

Substitution of Eq.(B2) into Eq.(B1), after observing that the operator $\exp(-\Delta\tau\partial_t)$ is equivalent to the time translation $t \rightarrow t - \Delta\tau$, one obtains:

$$h_L(t) = g_0 \mathcal{B}(i\partial_t) h_0(t - \Delta\tau) \quad (\text{B4})$$

where we have introduced the "complex" group delay:

$$\Delta\tau \equiv -i \left(\frac{\partial \ln g}{\partial \Omega} \right)_{\Omega_0} = \tau_g(\Omega_0) - i \left(\frac{\partial \ln \sqrt{G}}{\partial \Omega} \right)_{\Omega_0}. \quad (\text{B5})$$

From Eq.(B3), the following asymptotic expansion holds for the operator $\mathcal{B}(i\partial_t)$:

$$\mathcal{B}(i\partial_t) = 1 - \frac{1}{2} \left(\frac{\partial^2 \ln g}{\partial \Omega^2} \right)_{\Omega_0} \frac{\partial^2}{\partial t^2} + \dots \quad (\text{B6})$$

where the dots involve higher-order time derivatives; substitution of Eq.(B6) into Eq.(B4) finally yields:

$$h_L(t) = g_0 h_0(t - \Delta\tau) - \frac{g_0}{2} \left(\frac{\partial^2 \ln g}{\partial \Omega^2} \right)_{\Omega_0} \frac{\partial^2 h_0}{\partial t^2}(t - \Delta\tau) + \dots \quad (\text{B7})$$

At the leading order in the expansion, Eq.(B7) reduces to Eq.(25) given in the text.

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Figure Captions

Fig.1 Schematic of parametric amplification of a weak signal wave at frequency ω_1 in a QPM nonlinear crystal pumped by a strong CW pump at frequency ω_3 (copropagating interaction).

Fig.2 PPLN crystal geometry. In (a): the crystal comprises two uniform grating sections, each of length a , separated by a distance l with no grating structure. In each grating section, a $+/-$ square periodic wave pattern of domains with period Λ is assumed. This yields $d_{eff} = (2/\pi)d_{33}$ in the first grating and $d_{eff} = \exp(i\Phi)(2/\pi)d_{33}$ in the second grating, where the phase Φ is determined by the relative phase shift of square waves in the two grating sections. In (b) it is shown the QPM square wave profile $\chi^{(2)}(z)$ that corresponds to $\Phi = \pi$.

Fig.3 Behavior of spectral power gain (a) and group delay (b) for $\Phi = \pi$. The pump intensity is $I_3 = 135 \text{ MW/cm}^2$. The other parameter values are given in the text. Solid and dashed curves refer to the results obtained by use of Sellmeier equations, i.e. taking into account dispersion at any order, and by use of approximate equations (22) given in the text, respectively. The inset in (b) shows an enlargement of the group delay near resonance.

Fig.4 Same as Fig.3, but for $\Phi = \pi/2$.

Fig.5 Same as Fig.3, but for $\Phi = 0$.

Fig.6 Behavior of group delay τ_g versus pump intensity I_3 at the center of the amplifier gain dip. Parameter values are the same as in Fig.3.

Fig.7 Traces of the intensity of transmitted signal pulse (solid curves) at the exit plane $z = L$ of the amplifier for a few values of pump intensity. Curve 1: $I_3 = 0$; curve 2: $I_3 = 108 \text{ MW/cm}^2$; curve 3: $I_3 = 135 \text{ MW/cm}^2$; curve 4: $I_3 = 162 \text{ MW/cm}^2$. The dashed curve is the trace of incident Gaussian pulse at the input plane $z = 0$.

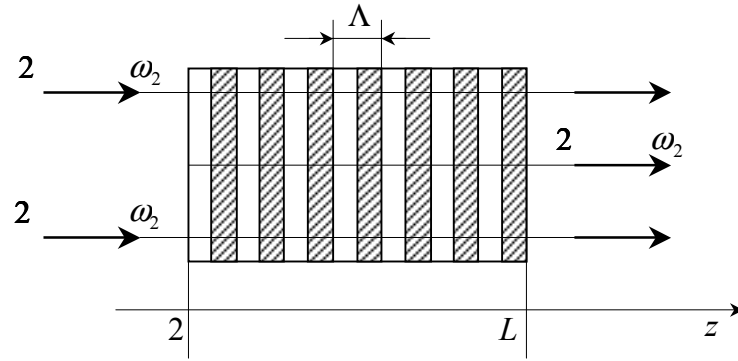


FIG. 1: S. Longhi et al., "Dispersive properties of ..."

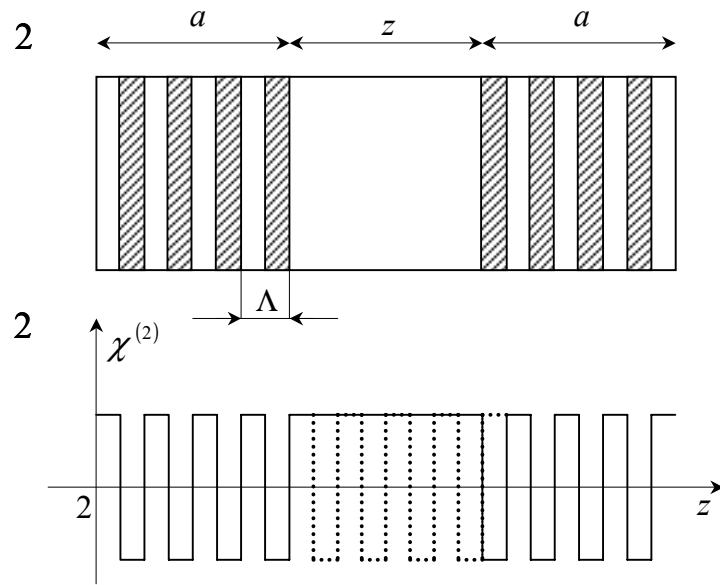


FIG. 2: S. Longhi et al., "Dispersive properties of ..."

